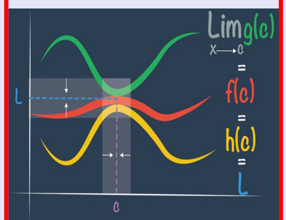
Math 261

Spring 2021

Lecture 6



Find $\lim\limits_{x \to 0} \dfrac{\sqrt{x+3}-\sqrt{3}}{x} = \dfrac{\sqrt{0+3}-\sqrt{3}}{0} = \dfrac{\sqrt{3}-\sqrt{3}}{0} = \dfrac{0}{0}$

I.F.

$$= \lim_{x \to 0} \frac{\overset{A}{\sqrt{x+3}} - \overset{B}{\sqrt{3}}}{x} \cdot \frac{\overset{A}{\sqrt{x+3}} + \overset{B}{\sqrt{3}}}{\sqrt{x+3}+\sqrt{3}}$$

$(A-B)(A+B) = A^2 - B^2$

$$= \lim_{x \to 0} \frac{x \cancel{+3} \cancel{-3}}{x(\sqrt{x+3}+\sqrt{3})} = \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+3}+\sqrt{3})}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+3}+\sqrt{3}}$$

$$= \frac{1}{\sqrt{0+3}+\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}} = \boxed{\frac{\sqrt{3}}{6}}$$

Consider $f(x) = \begin{cases} x^2-5 & \text{if } x \le 3 \\ \sqrt{x+13} & \text{if } x > 3 \end{cases}$

x^2-5 $\sqrt{x+13}$

$\rightarrow 3 \leftarrow$

1) $\lim_{x \to 3^-} f(x) = (3)^2-5 = \boxed{4}$

2) $\lim_{x \to 3^+} f(x) = \sqrt{3+13} = \boxed{4}$

3) $\lim_{x \to 3} f(x) = \boxed{4}$

one-sided limits are equal.

4) $f(3) = 3^2-5 = \boxed{4}$

5) $\underbrace{\text{Is } f(x) \text{ cont. at } x=3?}_{\text{Yes}}$ explain.

1) $\lim_{x \to 3} f(x) = 4$ ✓

2) $f(3) = 4$ ✓

3) $\lim_{x \to 3} f(x) = f(3)$ ✓

Prove $\lim_{x \to 3} (4x-5) = 7$ by finding a relationship between ε & δ.

$f(x) = 4x-5$ $L = 7$ $a = 3$

1) Verify $\lim_{x \to 3} (4x-5) = 4(3)-5 = 12-5 = 7$ ✓

For any $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

$|4x-5-7| < \varepsilon$ " $|x-3| < \delta$

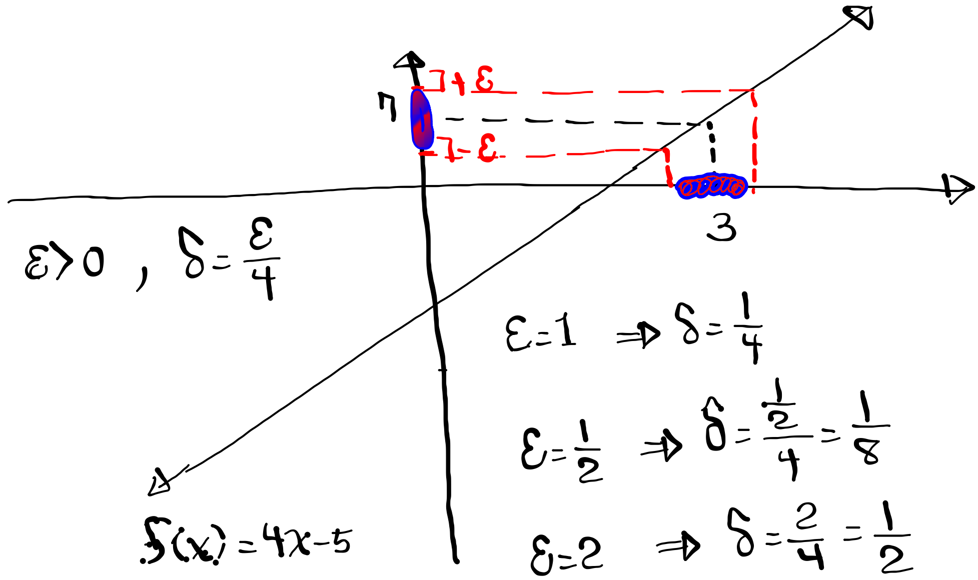
$|4x-12| < \varepsilon$ " $|x-3| < \delta$

$|4(x-3)| < \varepsilon$ " $|x-3| < \delta$

$4|x-3| < \varepsilon$ " $|x-3| < \delta$

$\boxed{|x-3|} < \frac{\varepsilon}{4}$ " $\boxed{|x-3|} < \delta$

Pick $\delta = \frac{\varepsilon}{4}$



For $\varepsilon > 0$, $\delta = \frac{\varepsilon}{4}$

$f(x) = 4x - 5$

$\varepsilon = 1 \Rightarrow \delta = \frac{1}{4}$

$\varepsilon = \frac{1}{2} \Rightarrow \delta = \frac{\frac{1}{2}}{4} = \frac{1}{8}$

$\varepsilon = 2 \Rightarrow \delta = \frac{2}{4} = \frac{1}{2}$

Prove $\lim_{x \to -2} (2 - 5x) = 12$, then find δ for $\varepsilon = .01$.

$f(x) = 2 - 5x$ $\quad L = 12$ $\quad a = -2$

1) Verify $\lim_{x \to -2} (2 - 5x) = 12$ ✓

$\lim_{x \to -2} (2 - 5x) = 2 - 5(-2) = 2 + 10 = 12$ ✓

For $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|2 - 5x - 12| < \varepsilon$ " $|x - (-2)| < \delta$

$|-5x - 10| < \varepsilon$ " $|x + 2| < \delta$

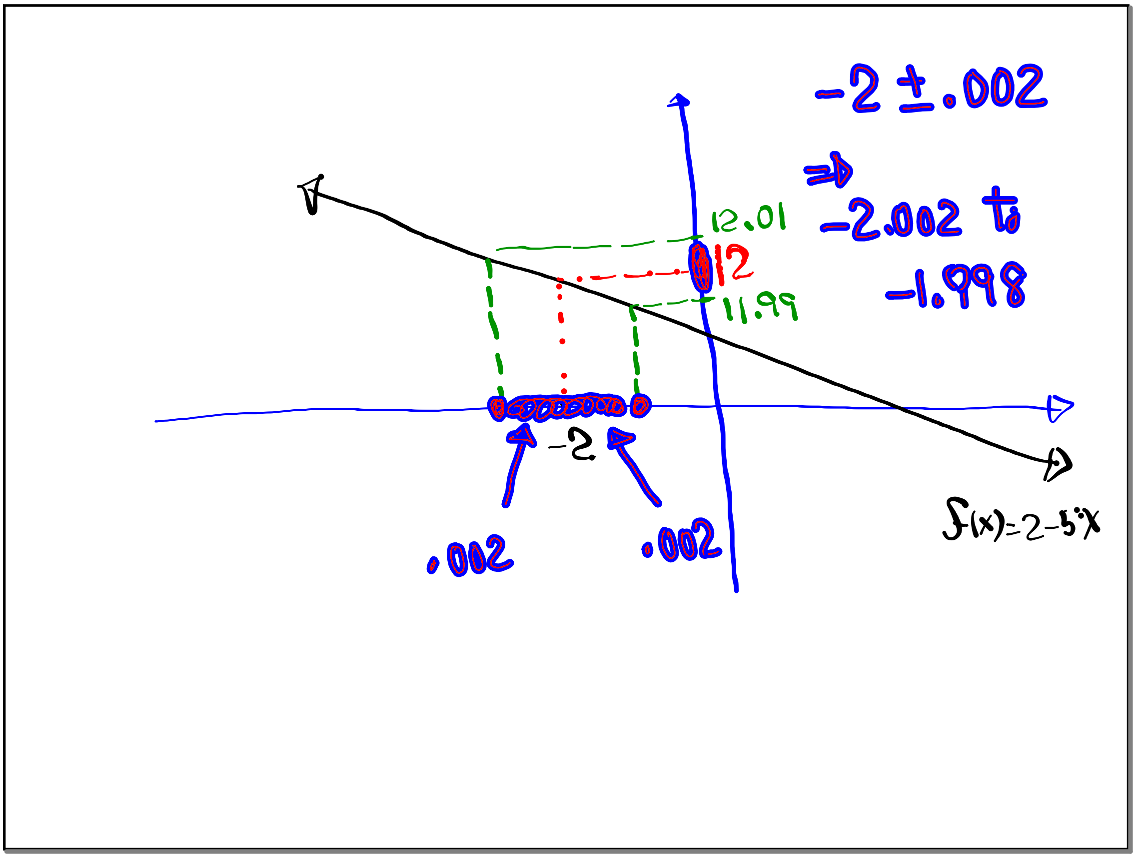
$|-5(x + 2)| < \varepsilon$ " $|x + 2| < \delta$

$5 \Rightarrow |-5|\,|x + 2| < \varepsilon$ " $|x + 2| < \delta$

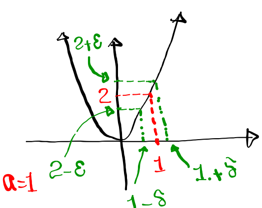
$|x + 2| < \frac{\varepsilon}{5}$ $\quad |x + 2| < \delta$

Pick $\delta = \frac{\varepsilon}{5}$

$\boxed{\text{For } \varepsilon = .01} \Rightarrow \delta = \frac{.01}{5} \Rightarrow \boxed{\delta = .002}$



$$\lim_{x\to 1} 2x^2 = 2$$



1) Verify ?

2) $f(x)=2x^2$ $L=2$ $a=1$

For $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|2x^2-2|<\varepsilon$

$|2(x^2-1)|<\varepsilon$

$2|(x+1)(x-1)|<\varepsilon$

$\underbrace{2|x+1|}_{\text{Bound}}\ \underbrace{|x-1|}_{\text{Keep}}<\varepsilon$

$|x-1|<\delta$

Let's agree that $\delta<1$

$|x-1|<1$

$-1<x-1<1$

$0<x<2$

$1<x+1<3$

$|x+1|<3$

$2|x+1|<6$

$2|x+1||x-1| < 6|x-1| < \varepsilon$

$|x-1| < \frac{\varepsilon}{6}$ ← δ

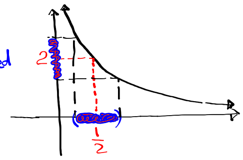
$$\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$$

$\varepsilon=1 \Rightarrow \delta=\frac{1}{6}$, $\varepsilon=2 \Rightarrow \delta=\frac{1}{3}$, $\varepsilon=8 \Rightarrow \delta=\cancel{\frac{8}{6}}$

$\Rightarrow \delta=1$

$$\lim_{x \to \frac{1}{2}} \frac{1}{x} = 2$$ undefined at 0

1) Verify? ✓



Let $\delta < \frac{1}{4}$

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$\left|\frac{1}{x} - 2\right| < \varepsilon$ $\quad$ $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{1 - 2x}{x}\right| < \varepsilon$

$\left|\frac{-2x + 1}{x}\right| < \varepsilon$ $\rightarrow$ $\left|\frac{-2(x - \frac{1}{2})}{x}\right| < \varepsilon$

$\underbrace{\left|\frac{-2}{x}\right|}_{\text{Bound}} \underbrace{\left|x - \frac{1}{2}\right|}_{\text{Keep}} < \varepsilon$

we agreed $\delta < \frac{1}{4}$

$\left|x - \frac{1}{2}\right| < \frac{1}{4}$ $\quad$ $-\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4}$

$\frac{1}{4} < x < \frac{3}{4}$

$4 > \frac{1}{x} > \frac{4}{3}$

$\left|\frac{-2}{x}\right| \left|x - \frac{1}{2}\right| < 8\left|x - \frac{1}{2}\right| < \varepsilon$ $\quad$ $\frac{4}{3} < \frac{1}{x} < 4$

$\boxed{\frac{8}{3} < \frac{2}{x} < 8}$

$\left|x - \frac{1}{2}\right| < \frac{\varepsilon}{8} \leftarrow \delta$

$\left|\frac{2}{x}\right| < 8$

$\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{8}\right\}$

Class QUIZ 2

1) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

2) $\lim_{x \to 0} \sqrt{x}$